Linearization of the Relativistic Oscillator Hierarchy

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Abstract

This paper is based on MacColl's [1] solution of the equation of motion for a linear (harmonic) oscillator subject to the laws of special relativity in the rest frame of the center of attraction. MacColl's result can be extended to the quartic oscillator in this frame with one extremely simple adjustment of the linearization map given in Anderson [2]. In fact, it can be extended to all the attractive oscillators in this frame.

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Part I

INTRODUCTION

MacColl in [1] solved the equation of motion for a linear harmonic oscillator subject to the laws of special relativity in the rest frame of the center of attraction.

In particular, he solved the equation of motion

$$\frac{d}{d\hat{t}} \frac{m_0 \frac{dx}{d\hat{t}}}{[1-(\frac{dx}{d\hat{t}})^2/c^2]^{1/2}} + k_2\, x = 0. \tag{1.1}$$

Before citing his solution, we set some notation with those in this paper and MacColl's.

$$\hat{t} = t_{MacColl}, \frac{x}{c} = \frac{x_{max}}{c} \, sin\, \theta = y_{MacColl}, \frac{x_{max}}{c} = \frac{\sqrt{2}}{\omega} (W-1)^{1/2},$$
$$k_2 = k_{MacColl} = m_0\, \omega^2, W = E/m_0\, c^2.$$

MacColl derived from (1.1), the following expression for the total energy E

$$E = m_0\, c^2\, (1-(\frac{dx}{d\hat{t}}/c)^2)^{-1/2} + \frac{1}{2} k_2\, x^2. \tag{1.2}$$

MacColl solved the equation of motion (1) in terms of elliptic functions

$$\hat{t} = \frac{\sqrt{2}}{\omega} (W+1)^{1/2} \int_0^\theta [\, 1 - \frac{W-1}{W+1} \sin^2\theta\,]^{1/2}\, dt$$
$$-\frac{\sqrt{2}}{\omega} (W+1)^{1/2} \int_0^\theta [\, 1 - \frac{W-1}{W+1} \sin^2\theta\,]^{-1/2}\, dt, \tag{1.3}$$

where the first integral in (1.3) is the incomplete elliptic integral of the second kind and the second integral in (1.3) is the incomplete integral of the first kind. For each θ, one obtains by quadrature $\hat{t}$ from (1.3) and $x(\hat{t})$ by

$$x(\hat{t}) = \frac{1}{\omega}\ \sqrt{2}\,(W-1)^{1/2} \sin\theta, \tag{1.4}$$

where $x_{\max} = \frac{1}{\omega}\ \sqrt{2}\,(W-1)$. We build on MacColl's impressive work.

In the Part II, 2 Linearization for the Quartic Oscillator, we shall with an extremely simple adaptation, apply the linearization map given in Anderson [2] to the correspondence between (1.1) and the quadratic potential equation of motion.

In the Part III, 3 Linearization Mapping for $\frac{1}{2n}\, k_{2n}\, y_{2n}^{2n}\,(t)|_{n>1}$ Hierarchy, we parallel the development in paragraph 2.

Part II

Linearization Map for the Quartic Oscillator

The linearization map in [2] implements the correspondence between the solutions to Newton's equations of motion for the harmonic oscillator *ho* and the quartic oscillator *qo*,

$$m\,\frac{d^2}{d^2\hat{t}}\,x(\hat{t}) + k_2\,x(\hat{t}) \;=\; 0 \;\;\Leftrightarrow\;\; m\,\frac{d^2}{d^2t}\,y(t) + k_4 y^3(t) = 0. \tag{2.1}$$

Note both systems are assumed to have the same mass m. Now, with an extremely simple adaptation, namely set m = m_0 = rest mass, this linearization implements the correspondence between

$$\frac{d}{d\hat{t}}\,\frac{m_0\frac{dx}{d\hat{t}}}{(1-(\frac{dx}{d\hat{t}}/c)^2)^{1/2}}\,+\,k_2\,x\,=\,0 \leftrightarrow \frac{d}{dt}\,\frac{m_0\frac{dy}{dt}}{(1-((\frac{dy}{d\hat{t}})/c)^2)^{1/2}}\,+\,k_2\,y\,=\,0. \tag{2.2}$$

The invertible linearization map to the quartic oscillator with rest mass m_0 and space coordinate y is stated in two parts. First,

$$y=(2k_2/k_4)^{1/4}x/(x^2)^{1/4} \tag{2.3a}$$

or

$$x=(k_4/2k_2)^{1/2}(y^2)^{1/2}y, \tag{2.3b}$$

where y is the space coordinate of the quartic oscillator and we have used the representation sgn(x) = $x/(x^2)^{1/2}$ and similarly for sgn(y). This implements the physical requirement that $\frac{1}{2}k_2x^2(\hat{t})$ = $\frac{1}{4}k_4y^4(t)$ i.e. matching the potential energies at the two different times, coupled with matching of the signs of the space coordinates. One cycle of the *qo* corresponds to one cycle of the *ho*, of course the periods are different.

Second,

$$\frac{dt}{d\hat{t}}=1/2(2k_2/k_4)^{1/4}(x^2(\hat{t}))^{-1/4} \tag{2.4a}$$

and

$$\frac{d\hat{t}}{dt}=(2k_2/k_4)^{1/2}(y^2(t))^{1/2}, \tag{2.4b}$$

which results by requiring

$$dx(\hat{t})/d\hat{t}=dy(t)/dt. \tag{2.5}$$

Specifically, in the fixed frame of the attractive center we have MacColl's (1.1). Invoking that m_0 is common and (2.3b), (2.4b), we obtain:

$$\frac{1}{(\frac{2k_4}{k_2})^{1/2}}\,\frac{1}{(y^2(t))^{1/2}}\,\frac{d}{dt}\frac{m_0\frac{dy}{dt}}{(1-(\frac{dy}{dt})^2/c^2)^{1/2}} + k_2\,(\frac{k_4}{2k_2})^{1/2}\,(y^2(t))^{1/2}\,y(t) = 0, \tag{2.6}$$

or

$$\frac{d}{dt}\Big(\frac{m_0\frac{dy}{dt}}{(1-(\frac{dy}{dt})^2/c^2)^{1/2}}\Big) + \underbrace{(k_4)^{1/2}\,(y^2(t))^{1/2}\,(y^2(t))^{1/2}\,y(t)}_{k_4\,y^2(t)\,y(t)} = 0. \tag{2.7}$$

Here in outline form is how to apply our linearization map for the special relativity oscillators in the rest frame of the center of attraction to obtain the solution to the quartic oscillator.

Step 1: Common in our construction to both the *ho* and the *qo*, we set the rest mass m_0, $\hat{t}_0 = t_0 = 0$ and E by selecting x_{max} given by (1.4).

Step 2: For the linear system, pick k_2. This sets ω by $k_2 = m_0\ \omega^2$.

Step 3: The correspondence $\hat{t} \leftrightarrow x(\hat{t})$ is set by MacColl's solution.

Step 4: For *qo*, select k_4.

Step 5: From (2.4a), we obtain

$$t - t_o = \int_{\hat{t}_o}^{\hat{t}} \frac{1}{2} \left(\frac{2k_2}{k_4}\right)^{1/4} (x^2(\hat{t}'))^{-1/4} \, d\hat{t}', \tag{2.8}$$

which is the quadrature with known integrand in the approach given in [1]. This sets t (including t_a and t_b).

Step 6:

Now from $\frac{1}{2}\, k_2\, x^2(\hat{t}) = \frac{1}{4}\, k_4\, y^4(t)$, we obtain from (2.3a)

$$y\,(t) = \left(\frac{2k_2}{k_4}\right)^{1/4} (x^2(\hat{t}))^{-1/4}\, x(\hat{t}), \tag{2.9}$$

where $\hat{t}$ is the upper limit *ho* time used in (2.8).

Part III

Linearization Mapping for $\frac{1}{2n} k_{2n} y_{2n}^{2n}(t)|_{n>1}$ Hierarchy

In a straight forward manner the mappings in Part II, generalize and yield the following relationships:

(A)

$$y_{2n} = (nk_2/k_{2n})^{1/2n}\, x(x^2)^{(1/2)(1-n)/n} \tag{3.1a}$$

$$x = (k_{2n}/nk_2)^{1/2}\, y_{2n}(y_{2n}^2)^{(n-1)/2}, \tag{3.1b}$$

which is the generalizations of (2.3a) and (2.3b), respectively. The generalization of (2.4a) and (2.4b), respectively is given by:

(B)

$$\frac{dt_{2n}}{d\hat{t}} = n^{-(2n-1)/2n}(k_2/k_{2n})^{1/2n}(x^2(\hat{t}))^{-(n-1)/2n}(\hat{t}), \tag{3.2a}$$

and

$$\frac{d\hat{t}}{dt_{2n}} = \sqrt{n}(k_{2n}/k_2)^{1/2}(y_{2n}^2)^{(n-1)/2}. \tag{3.2b}$$

These mappings take the space-time extremals of the linear oscillator with coordinates (x, $\hat{t}$) and map them onto the space-time extremals of the $2n^{\text{th}}$ oscillator with coordinates (y_{2n}, t).

This linearization implements the correspondence between

$$\frac{d}{d\hat{t}}\ \frac{m_0 \frac{dx}{d\hat{t}}}{\sqrt{1-(\frac{dx}{d\hat{t}}/c)^2}} + k_2 x(\hat{t}) = 0 \Leftrightarrow \frac{d}{dt_{2n}}\ (\ m_0\, dy_{2n}\ /\ dt_{2N}\ /\ (\ 1-(\ \frac{dy_{2n}}{dt_{2n}}/\ c\ )^2)^{1/2}\ )\ +\ k_{2n}\, y_{2n}^{2n-1} = 0. \tag{3.3}$$

A straightforward calculation paralleling that in Section II yields

Further, as a consequence of the above, we have the following equality for the conserved total energies

$$E_2 = E_{2n}, \tag{3.4}$$

Specifically, in the fixed frame of the attractive center, we have, invoking m_0 is common and applying (3.1b) and (3.2b), we have that

$$\frac{1}{\sqrt{n}\,(\frac{k_{2n}}{k_2})^{1/2}(y_{2n}^2)^{(n-1/2)}}\ \frac{d}{dt_{2n}}\left(\frac{m_0\, dy\ /\ 2n\ /dt_{2n}}{(1-(\frac{dy_{2n}}{dt_{2n}}/c)^2)^{1/2}}\right) + k_2\left(\frac{k_{2n}}{nk_2}\right)^{1/2} y_{2n}(y_{2n}^2)^{(n-1)/2} = 0, \tag{3.5}$$

which yields the rhs of the correspondence (3.3).

The operative deformation of time given by (3.2a) becomes in integral form

$$t - t_0 \ = \ \int_{\hat{t}_0}^{\hat{t}} n^{-(2n-1)/2n} (k_2/k_{2n})^{1/2n} (x^2(\hat{t}))^{-(n-1)/2n} d\hat{t}. \tag{3.6}$$

All of the analyses presented in Part II can then be paralleled to obtain the members of the hierarchy. Note $y_2 = y$.

References

[1] L.A. MacColl, ”Theory of the Relativistic Oscillator”, Amer. J. Physics, 25, 535 (1957).

[2] Robert L. Anderson, ”An Invertible Linearization Map for the Quartic Oscillator”, JMP, 51, 122904 (2010).